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Analytic solutions to a class of nonlinear infinite-delay-differential equations ☆

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Abstract

The infinite-delay-differential equations (IDDEs) are studied and the analytic solution of a class of nonlinear IDDEs is presented based on the characteristics of the reproducing kernel space $W_2[0,\infty)$. Besides, the exact solution is represented in the form of series. It is proved that the n-term approximation $u_n(x)$ converges to the exact solution $u(x)$ of the IDDEs. Moreover, the approximate error of $u_n(x)$ is monotone decreasing. The results of experiments showed that the proposed method in this paper is computationally efficient.

Keywords: Exact solution; Infinite-delay-differential equation; Reproducing kernel space

1. Introduction

The infinite-delay-differential equation (IDDE) is a special case of the functional differential equation (FDE). Let us consider the general form of the nonlinear infinite-delay-differential equation (IDDE) with proportional delay:

$$\begin{cases} u'(x) = g\big(x, u(x), u(px)\big), \\ u(0) = \eta, \end{cases} \tag{1.1}$$

where $p \in (0,1)$, η is a given initial value, $u(x) \in W_2[0,+\infty)$, and for $x \in [0,+\infty)$, $y, z \in (-\infty,+\infty)$, $g(x,y,z)$ is continuous function; $g(x,y,z) \in W_1[0,+\infty)$ as $y = y(x)$, $z = z(x) \in W_1[0,+\infty)$; $W_2[0,+\infty)$ and $W_1[0,\infty)$ are reproducing kernel spaces.

It has been found that the characteristics of FDEs with proportional delays and those with constant delays are substantially different. So far, some research works on the numerical solutions and the corresponding analysis for the FDE with proportional delays have been presented in Refs. [1–6]. In recent years, some researches on the existence

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of solutions of FDEs with state dependent delay and their computational algorithms have been reported [7–15], and initial-value problems of neutral FDEs with proportional time delays have been studied in Refs. [16–19]. However, some problems for solving the IDDEs with proportional delays are still open [20–23]. Therefore, how to solve Eq. (1.1) in a reproducing kernel space is studied in this paper and a new method is presented.

Suppose L is an operator, let $Lu(x) \equiv u'(x)$, due to the homogenization of initial condition, Eq. (1.1) can be converted into the following equivalent form:

$$\begin{cases} Lu(x) = f\big(x, u(x), u(px)\big), \\ u(0) = 0, \end{cases} \tag{1.2}$$

where $x \in [0, +\infty)$, $u(x) \in W_2[0, +\infty)$ and for $x \in [0, +\infty)$, $y, z \in (-\infty, +\infty)$, $f(x, y, z)$ is continuous function; $f(x, y, z) \in W_1[0, +\infty)$ as $y = y(x)$, $z = z(x) \in W_1[0, +\infty)$.

2. Preliminaries

Let us introduce the definitions of two reproducing kernel spaces.

2.1. The reproducing kernel space $W_2[0, +\infty)$

Definition 2.1. $W_2[0, +\infty) = \{u(x) \mid u, u'$ are one-variable absolutely continuous real-valued functions on $[0, +\infty)$, $u'' \in L^2[0, +\infty), u(0) = 0\}$.

$W_2[0, +\infty)$ is a Hilbert space. For $u(x), v(x) \in W_2[0, +\infty)$, the inner product and norm in $W_2[0, +\infty)$ are given by the following:

$$\big\langle u(x), v(x)\big\rangle_{W_2} = \int_0^{+\infty} (4uv + 5u'v' + u''v'')\, dx, \qquad \|u\|_{W_2} = \sqrt{\langle u, u\rangle_{W_2}}, \tag{2.1}$$

respectively.

It has been proved that $W_2[0, +\infty)$ is a complete reproducing kernel space and the corresponding reproducing kernel can be represented as follows:

$$R_x(y) = \begin{cases} -\dfrac{1}{12} e^{-2(x+y)}\big(-1 + e^{2y}\big)\big(1 + e^{2y} - 2e^{x+y}\big), & y \leqslant x, \\ -\dfrac{1}{12} e^{-2(x+y)}\big(-1 + e^{2x}\big)\big(1 + e^{2x} - 2e^{x+y}\big), & y > x, \end{cases} \tag{2.2}$$

so, $\forall u(x) \in W_2[0, +\infty)$, we have $u(x) = \langle R_x(y), u(y)\rangle$ (see [24]).

2.2. The reproducing kernel space $W_1[0, +\infty)$

Definition 2.2. $W_1[0, +\infty) = \{u(x) \mid u$ is absolutely continuous real-valued function, $u' \in L^2[0, +\infty)\}$.

The inner product and norm in $W_1[0, +\infty)$ can be defined by

$$\big\langle u(x), v(x)\big\rangle_{W_1} = \int_0^{+\infty} (uv + u'v')\, dx, \qquad \|u\|_{W_1} = \sqrt{\langle u, u\rangle_{W_1}},$$

respectively, where $u(x), v(x) \in W_1[0, +\infty)$. It has been proved that $W_1[0, +\infty)$ is a complete reproducing kernel space and its reproducing kernel is as follows:

$$Q_x(y) = \begin{cases} \dfrac{1}{2} e^{-x-y}\big(1 + e^{2x}\big), & x < y, \\ \dfrac{1}{2} e^{-x-y}\big(1 + e^{2y}\big), & x \geqslant y. \end{cases} \tag{2.3}$$

Lemma 2.1. *If* $u(x) \in W_2[0, +\infty)$*, then* $\exists M_i > 0$ $(i = 0, 1)$*, such that* $\|u^{(i)}\|_C \leqslant M_i \|u\|_{W_2}$.

Proof. Let $R_x(y)$ be the reproducing kernel of $W_2[0, +\infty)$, for any $x, y \in [0, +\infty)$, it follows that

$$\begin{cases} u(x) = \langle u(y), R_x(y)\rangle_{W_2}, \\ u'(x) = \langle u(y), \partial_x R_x(y)\rangle_{W_2}. \end{cases}$$

Note that

$$\big\|\partial_x^{(i)} R_x(y)\big\|_{W_2} \leqslant M_i, \quad i = 0, 1,$$

hence we can get

$$\big|u^{(i)}(x)\big| = \big|\langle u(y), \partial_x^{(i)} R_x(y)\rangle\big|_{W_2} \leqslant \big\|u(y)\big\|_{W_2} \big\|\partial_x^{(i)} R_x(y)\big\|_{W_2} \leqslant M_i \|u\|_{W_2},$$

thus

$$\big\|u^{(i)}\big\|_C \leqslant M_i \|u\|_{W_2}, \quad i = 0, 1. \qquad \square$$

3. The solution of Eq. (1.2)

In this section, the solution of Eq. (1.2) is given in the reproducing kernel space $W_2[0, +\infty)$. By definitions of $W_1[0, +\infty)$ and $W_2[0, +\infty)$, it is clear that the operator $L : W_2[0, +\infty) \to W_1[0, +\infty)$ is a bounded linear operator. Let $\varphi_i(x) = Q_{x_i}(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[0, +\infty)$, and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the conjugate operator of L. For all $u(x) \in W_1[0, +\infty)$, we have

$$\langle u(x), \varphi_i(x)\rangle = u(x_i). \tag{3.1}$$

The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is derived from Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$, namely,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x) \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots). \tag{3.2}$$

Theorem 3.1. *For Eq.* (1.2)*, assume that* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0, +\infty)$*, then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2[0, +\infty)$ *and* $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof. We have

$$\psi_i(x) = \big(L^*\varphi_i\big)(x) = \big\langle\big(L^*\varphi_i\big)(y), R_x(y)\big\rangle = \big\langle\varphi_i(y), L_y R_x(y)\big\rangle = L_y R_x(y)|_{y=x_i},$$

where the subscript y of the operator L indicates that the operator L applies to the function of y. Obviously, $\psi_i(x) \in W_2[0, +\infty)$.

For each fixed $u(x) \in W_2[0, +\infty)$, let $\langle u(x), \psi_i(x)\rangle = 0$ $(i = 1, 2, \ldots)$, which means that

$$\big\langle u(x), \big(L^*\varphi_i\big)(x)\big\rangle = \big\langle Lu(\cdot), \varphi_i(\cdot)\big\rangle = (Lu)(x_i) = 0. \tag{3.3}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense in $[0, +\infty)$, hence $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}.

So the proof of Theorem 3.1 is complete. $\square$

Theorem 3.2. *Let* $\{x_i\}_{i=1}^{\infty}$ *be dense in* $[0, +\infty)$*, if* $u(x)$ *is unique solution of Eq.* (1.2)*, then* $u(x)$ *satisfies the form*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f\big(x_k, u(x_k), u(px_k)\big)\overline{\psi}_i(x). \tag{3.4}$$

Proof. Assume that $u(x)$ is the solution of Eq. (1.2). Applying Theorem 3.1, it is easy to know that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2[0,+\infty)$.

Note that (3.1) and (3.2), hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\left\langle u(x), \sum_{k=1}^{i}\beta_{ik}\psi_k(x)\right\rangle\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\big\langle u(x),\psi_k(x)\big\rangle\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\big\langle u(x), L^*\varphi_k(x)\big\rangle\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\big\langle Lu(x),\varphi_k(x)\big\rangle\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f\big(x_k, u(x_k), u(px_k)\big)\overline{\psi}_i(x). \qquad \square \end{aligned} \tag{3.5}$$

Let $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ be the normal orthogonal system derived from Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$, then the solution of Eq. (1.2) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} A_i\overline{\psi}_i(x), \tag{3.6}$$

where $A_i = \sum_{k=1}^{i}\beta_{ik} f(x_k, u(x_k), u(px_k))$. Since $f(x_1, u(x_1), u(px_1))$ is known as $x_1 = 0$, considering the numerical computation, we put $u_0(x_1) = u(x_1) = 0$, and define the n-term approximation $u_n(x)$ to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n}\overline{A}_i\overline{\psi}_i(x), \tag{3.7}$$

where

$$\begin{cases} \overline{A}_1 = \beta_{11} f\big(x_1, u_0(x_1), u_0(px_1)\big), \\ \overline{A}_2 = \displaystyle\sum_{k=1}^{2}\beta_{2k} f\big(x_k, u_{k-1}(x_k), u_{k-1}(px_k)\big), \\ \vdots \\ \overline{A}_n = \displaystyle\sum_{k=1}^{n}\beta_{nk} f\big(x_k, u_{k-1}(x_k), u_{k-1}(px_k)\big). \end{cases} \tag{3.8}$$

To prove the convergence of $u_n(x)$, the lemma is required as follows.

Lemma 3.1. *If $u_n(x) \xrightarrow{\|\cdot\|_{W_2}} \bar{u}(x)$ $(n\to+\infty)$, $x_n \to y$ $(n\to+\infty)$, $\|u_n(x)\|$ is bounded, and $f(x,y,z)$ is continuous as $x\in[0,+\infty)$, $y=y(x)$, $z=z(x)\in(-\infty,+\infty)$, then*

$$f\big(x_n, u_{n-1}(x_n), u_{n-1}(px_n)\big) \to f\big(y, \bar{u}(y), \bar{u}(py)\big) \quad \text{as } n\to+\infty.$$

Proof. Since $u_n(x) \xrightarrow{\|\cdot\|_{W_2}} \bar{u}(x)$ $(n\to+\infty)$, by Lemma 2.1, $u_n(x)$ converges uniformly to $\bar{u}(x)$ $(n\to+\infty)$. $\square$

Theorem 3.3. *Under the conditions of Theorem 3.2, suppose that $\|u_n(x)\|$ is bounded, $f(x,y,z)\in W_1[0,+\infty)$ as $y=y(x)$, $z=z(x)\in W_1[0,+\infty)$, then the n-term approximate solution $u_n(x)$ derived from (3.7) converges to the exact solution $u(x)$ of Eq. (1.2) and $u(x)=\sum_{i=1}^{\infty}\overline{A}_i\overline{\psi}_i(x)$, where $\overline{A}_i$ is given by (3.8).*

Proof. (1) First, we should prove the convergence of $u_n(x)$.

By (3.7), we infer that

$$u_{n+1}(x) = u_n(x) + \overline{A}_{n+1}\overline{\psi}_{n+1}(x). \tag{3.9}$$

From the orthogonality of $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned}\|u_{n+1}\|^2_{W_2[0,+\infty)} &= \|u_n\|^2_{W_2[0,+\infty)} + (\overline{A}_{n+1})^2 \\ &= \|u_{n-1}\|^2_{W_2[0,+\infty)} + (\overline{A}_n)^2 + (\overline{A}_{n+1})^2 \\ &\;\vdots \\ &= \|u_0\|^2_{W_2[0,+\infty)} + \sum_{i=1}^{n+1}(\overline{A}_i)^2. \end{aligned} \tag{3.10}$$

From (3.10), it is easy to see that sequence $\|u_n(x)\|_{W_2[0,+\infty)}$ is monotone increasing. Due to the boundedness of $\|u_n(x)\|_{W_2[0,+\infty)}$, $\|u_n(x)\|_{W_2[0,+\infty)}$ is convergent as $n \to \infty$ and

$$\sum_{i=1}^{\infty}(\overline{A}_i)^2 < \infty.$$

That implies that

$$\overline{A}_i = \sum_{k=1}^{i} \beta_{ik} f\bigl(x_k, u_{k-1}(x_k), u_{k-1}(px_k)\bigr) \in l^2 \quad (i = 1, 2, \ldots). \tag{3.11}$$

Without loss of generality, assuming $m > n$, we have

$$\begin{aligned}\bigl\|u_m(x) - u_n(x)\bigr\|^2_{W_2[0,+\infty)} &= \bigl\|u_m(x) - u_{m-1}(x) + u_{m-1}(x) - u_{m-2}(x) + \cdots + u_{n+1}(x) - u_n(x)\bigr\|^2_{W_2[0,+\infty)} \\ &\leqslant \bigl\|u_m(x) - u_{m-1}(x)\bigr\|^2_{W_2[0,+\infty)} + \cdots + \bigl\|u_{n+1}(x) - u_n(x)\bigr\|^2_{W_2[0,+\infty)} \\ &= \sum_{i=n+1}^{m} (\overline{A}_i)^2 \to 0 \quad (n \to +\infty).\end{aligned}$$

Considering the completeness of $W_2[0, +\infty)$, there exists $\bar{u}(x) \in W_2[0, +\infty)$ such that

$$u_n(x) \xrightarrow{\|\cdot\|} \bar{u}(x) \quad \text{as } n \to +\infty.$$

(2) Second, let us prove that $\bar{u}(x)$ is the solution of Eq. (1.2).

By Lemma 3.1 and (1) of Theorem 3.3, we know $u_n(x)$ converges uniformly to $\bar{u}(x)$ $(n \to \infty)$. It follows that, by setting limits for n in (3.7), we have

$$\bar{u}(x) = \sum_{i=1}^{\infty} \overline{A}_i \overline{\psi}_i(x).$$

Since

$$(L\bar{u})(x_j) = \sum_{i=1}^{\infty} \overline{A}_i \langle L\overline{\psi}_i, \varphi_j\rangle = \sum_{i=1}^{\infty} \overline{A}_i \bigl\langle \overline{\psi}_i, L^*\varphi_j\bigr\rangle = \sum_{i=1}^{\infty} \overline{A}_i \langle \overline{\psi}_i, \psi_j\rangle,$$

hence

$$\sum_{j=1}^{n} \beta_{nj} (L_1\bar{u})(x_j) = \sum_{i=1}^{\infty} \overline{A}_i \left\langle \overline{\psi}_i, \sum_{j=1}^{n} \beta_{nj}\psi_j \right\rangle = \sum_{i=1}^{\infty} \overline{A}_i \langle \overline{\psi}_i, \overline{\psi}_n\rangle = \overline{A}_n.$$

If $n = 1$, then

$$(L\bar{u})(x_1) = f\bigl(x_1, u_0(x_1), u_0(px_1)\bigr). \tag{3.12}$$

If $n = 2$, then

$$\beta_{21}(L\bar{u})(x_1) + \beta_{22}(L\bar{u})(x_2) = \beta_{21} f\bigl(x_1, u_0(x_1), u_0(px_1)\bigr) + \beta_{22} f\bigl(x_2, u_1(x_2), u_1(px_2)\bigr). \tag{3.13}$$

From (3.12) and (3.13), it is clear that

$$(L\bar{u})(x_2) = f\big(x_2, u_1(x_2), u_1(px_2)\big).$$

Furthermore, it is easy to see by induction that

$$(L\bar{u})(x_j) = f\big(x_j, u_{j-1}(x_j), u_{j-1}(px_j)\big), \quad j = 1, 2, \ldots. \tag{3.14}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[0, \infty)$, $\forall y \in [0, \infty)$, there exists a subsequence $\{x_{n_j}\}$ such that

$$x_{n_j} \to y \quad \text{as } j \to \infty.$$

Hence, let $j \to \infty$ in (3.14), by the convergence of $u_n(x)$ and Lemma 3.1, we have

$$(L\bar{u})(y) = f\big(y, \bar{u}(y), \bar{u}(py)\big). \tag{3.15}$$

Since $\overline{\psi}_i(M) \in W_2[0, +\infty)$, clearly, $\bar{u}(x)$ satisfies the initial condition of Eq. (1.2). That means that $\bar{u}(x)$ is the solution of Eq. (1.2).

The application to the uniqueness of solution to Eq. (1.2) yields that

$$u(x) = \sum_{i=1}^{\infty} \overline{A}_i \overline{\psi}_i(x). \tag{3.16}$$

The proof is complete. □

Theorem 3.4. *Assume that $u(x)$ is the solution of Eq.* (1.2), *r_n is the approximate error of $u_n(x)$, and $r_n = \|u(x) - u_n(x)\|_{W_2}$, where $u_n(x)$ is given by* (3.7). *Then the error r_n is monotone decreasing in the sense of $\|\cdot\|_{W_2[0,+\infty)}$.*

Proof. From (3.7) and (3.16), it follows that

$$\big\|r_n(x)\big\|_{W_2}^2 = \left\| \sum_{i=n+1}^{\infty} \overline{A}_i \overline{\psi}_i(x) \right\|_{W_2}^2 = \sum_{i=n+1}^{\infty} (\overline{A}_i)^2. \tag{3.17}$$

(3.17) shows that the error r_n is monotone decreasing in the sense of $\|\cdot\|_{W_2[0,+\infty)}$. □

4. Numerical example

The purpose of this section is to present some examples to illustrate computationally the results established in the paper. In the process of computation, all the symbolic and numerical computations performed by using Mathematica 5.0.

Example. Let us consider the following equation:

$$\begin{cases} u'(x) = xu^2\left(\dfrac{x}{2}\right) + u(x) + f(x), \\ u(0) = 0 \end{cases}$$

where $f(x) = 4x^2 + 9(-1 + e^{-1+x}) + 9e^{-1+x}x - 9(-1 + e^{-1+x})x - 4x^2 - \frac{81}{4}(-1 + e^{-1+\frac{x}{2}})^2 x^3$. The true solution is $u(x) = 9x(e^{x-1} - 1)$. Using our method, we only choose 100 points in [0, 1] and 200 points in [0, 5], and calculate the relative errors of $\|u^{(k)}(x) - u_n^{(k)}(x)\|_C$ $(k = 0, 1)$, respectively. The numerical results and errors are reported in Table 1 and in Figs. 1, 2, which show the method in the paper is efficient.

Table 1
Relative errors of $\|u^{(k)}(x) - u_n^{(k)}(x)\|_C$ $(k = 0, 1)$

Node	$\|u(x) - u_n(x)\|_C$	$\|u'(x) - u'_n(x)\|_C$
1/100	6.1891E–04	6.1865E–04
1/25	6.40289E–04	6.34674E–04
9/100	6.76941E–04	6.48101E–04
4/25	7.28948E–04	6.36234E–04
1/4	7.93186E–04	5.62724E–04
9/25	8.59367E–04	3.74203E–04
49/100	9.04314E–04	1.15443E–07
16/25	8.8517E–04	6.37698E–04
81/100	7.34582E–04	3.5683E–04

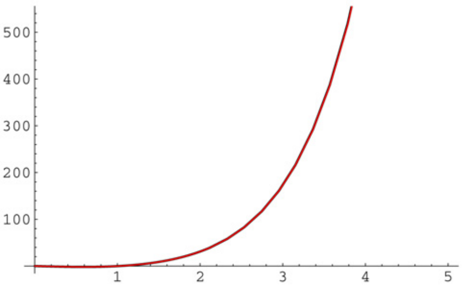

Fig. 1. The superimposed image of $u(x)$ with $u_{200}(x)$ in [0, 5].

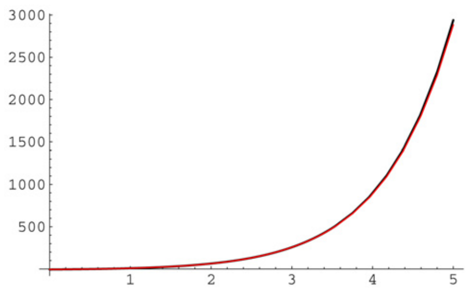

Fig. 2. The superimposed image of $u'(x)$ with $u'_{200}(x)$ in [0, 5].

Appendix A. The reproducing kernel space $W_2[0, m]$

$W_2[0, m]$ is defined as $W_2[0, m] = \{u(x) \mid u, u'$ are absolutely continuous real value functions, $u, u', u'' \in L^2[0, m]$, $u(0) = 0\}$. The inner product in $W_2[0, m]$ is given by

$$\langle u(y), v(y)\rangle_{W_2} = \int_0^m (4uv + 5u'v' + u''v'')\,dy, \tag{A.1}$$

where $u, v \in W_2[0, m]$ and the norm $\|u\|_{W_2}$ is denoted by $\|u\|_{W_2} = \sqrt{\langle u, u\rangle}_{W_2}$.

Theorem A.1. *The space $W_2[0, m]$ is a reproducing kernel space, that is, for any $u(y) \in W_2[0, m]$ and each fixed $x \in [0, m]$, there exists $R_x(y) \in W_2[0, m]$, $y \in [0, m]$, such that $\langle u(y), R_x(y)\rangle_{W_2[0,m]} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y}, & y \leqslant x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y}, & y > x. \end{cases} \tag{A.2}$$

Proof. Applying the integrations by parts for (A.1), we have

$$\langle u(y), R_x(y)\rangle_{W_2[0,m]} = \int_0^m u(y)\big[4R_x(y) - 5R''_x(y) + R^{(4)}_x(y)\big]\,dy + 5u(y)R'_x(y)\big|_0^m + u'(y)R''_x(y)\big|_0^m - u(y)R^{(3)}_x(y)\big|_0^m.$$

Since $R_x(y) \in W_2[0, m]$, it follows that

$$R_x(0) = 0. \tag{A.3}$$

For $u(y) \in W_2[0, m]$, thus $u(0) = 0$.

Suppose that $R_x(y)$ satisfies the following generalized differential equations:

$$\begin{cases} 4R_x(y) - 5R_x''(y) + R_x^{(4)}(y) = \delta(y-x), \\ 5R_x'(m) - R_x^{(3)}(m) = 0, \\ R_x''(m) = 0, \\ R_x''(0) = 0. \end{cases} \tag{A.4}$$

Then $\langle u(y), R_x(y)\rangle_{W_2[0,m]} = \int_0^m u(y)\delta(y-x)\,dy = u(x)$. Hence, $R_x(y)$ is the reproducing kernel of space $W_2[0,m]$.

In the following, we will get the expression of the reproducing kernel $R_x(y)$.

The characteristic equation of $R_x(y) - 5R_x''(y) + R_x^{(4)}(y) = \delta(y-x)$ is given by $\lambda^4 - 5\lambda^2 + 4 = 0$, and the characteristic roots are $\lambda_{1,2} = \pm 1$, $\lambda_{3,4} = \pm 2$.

We denote $R_x(y)$ by

$$R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y}, & y \leqslant x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y}, & y > x. \end{cases} \tag{A.5}$$

By the definition of space $W_2[0,m]$, coefficients $c_1, \dots, c_4, d_1, \dots, d_4$ satisfy

$$\begin{cases} R_x^{(k)}(x+0) = R_x^{(k)}(x-0) \quad (k = 0,1,2), \\ R_x^{(3)}(m) - 5R_x'(m) = 0, \\ R_x^{(3)}(x+0) - R_x^{(3)}(x-0) = 1, \\ R_x''(m) = 0, \\ R_x''(0) = 0, \\ R_x(0) = 0, \end{cases} \tag{A.6}$$

from which, the unknown coefficients of (A.5) can be obtained.

$$c_1 = \frac{e^{-2x}(4e^{3m} - 7e^{3x} - 9e^{2m+x} + 7e^{6m+x} + 9e^{4m+3x} - 4e^{3m+4x})}{6(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$c_2 = \frac{e^{-2x}(-4e^{3m} + 7e^{3x} + 9e^{2m+x} - 7e^{6m+x} - 9e^{4m+3x} + 4e^{3m+4x})}{6(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$c_3 = \frac{e^{-2x}(-9e^{4m} - 7e^{6m} + 7e^{4x} + 8e^{3m+x} - 8e^{3(m+x)} + 9e^{2m+4x})}{12(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$c_4 = \frac{-e^{-2x}(-9e^{4m} - 7e^{6m} + 7e^{4x} + 8e^{3m+x} - 8e^{3(m+x)} + 9e^{2m+4x})}{12(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$d_1 = -\frac{e^{-2x}(-1 + e^{2x})(4e^{3m} + 7e^{x} - 9e^{4m+x} + 4e^{3m+2x})}{6(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$d_2 = \frac{e^{2m-2x}(-1 + e^{2x})(4e^{m} + 7e^{4m+x} - 9e^{x} + 4e^{m+2x})}{6(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$d_3 = \frac{e^{-2x}(-1 + e^{2x})(7 + 9e^{2m} + 7e^{2x} + 9e^{2(m+x)} - 8e^{3m+x})}{12(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})},$$

$$d_4 = -\frac{e^{3m-2x}(-1 + e^{2x})(9e^{m} + 7e^{3m} + 9e^{m+2x} - 8e^{x} + 7e^{3m+2x})}{12(-7 - 9e^{2m} + 9e^{4m} + 7e^{6m})}. \qquad \square$$

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